Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 5, pp. 873–888, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00113-8

THE EQUILIBRIUM SHAPE OF AN ICE–SOIL BODY FORMED BY LIQUID FLOW PAST A PAIR OF FREEZING COLUMNS†

M. M. ALIMOV, K. G. KORNEV and G. L. MUKHAMADULLINA

Moscow

(*Received* 2 *November* 1993)

The methods proposed in [1] are used to solve the two-dimensional problem of congealing flow in a porous medium around a system of two freezing columns. Equilibrium configurations for ice–soil body are found for a wide range of the governing physical parameters. The results are used to reveal the inconsistency of certain linking criteria.

In most applications [2, 3] the characteristic transverse dimensions of the frozen region are small compared to the longitudinal dimensions, and in the first approximation the problem of determining the shape of the ice–soil body can be formulated two-dimensionally. The specific feature of the problem is the possibility of the existence of a limiting equilibrium ice–soil body. Here the heat flux from the thawed background is balanced by conduction heat transfer to the column.

One of the main problems of artificial freezing is to find the conditions for the linking of ice–soil bodies formed around individual columns. A base estimate is obtained by solving the problem of two freezing columns. An approximate solution of this problem exists [2] based on modelling an individual ice–soil body by a circular cylinder, which gives a linking criterion that is several times larger than that observed in practice. Another well-known criterion [4] is constructed by asymptotically solving the problem for ice–soil bodies that have already merged. This is actually a "non-separation" criterion, but its validity has not been analysed.

1. STATEMENT OF THE PROBLEM

The process is assumed to take place in the $Z = X + iY$ plane (Fig. 1a), filtration obeys Darcy's law, the liquid is incompressible, the thermal properties of the porous medium are constant, and the mathematical model of the phenomenon under consideration can be represented in the dimensionless form

$$\mathbf{v} = -\nabla p, \quad \operatorname{div}\mathbf{v} = 0, \quad z \in D_z^-; \quad |\mathbf{v}| = 1, \quad |z| \to \infty \tag{1.1}$$

$$\mathrm{Pe}(\mathbf{v}\nabla\theta^-) = \Delta\theta^-, \quad z \in D_z^-; \quad \theta^- = 1, \quad |z| \to \infty \tag{1.2}$$

$$\Delta\theta^+ = 0, \quad z \in D_z^+; \quad \partial\theta^+ / \partial n = \partial\theta^- / \partial n, \quad \theta^+ = \theta^- = 0, \quad z \in \Gamma_z \tag{1.3}$$

$$\lim_{r \to 0} r\partial\theta^+ / \partial r = q, \quad z = z_k \quad (k = 1, 2) \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 5, pp. 110–124, 1994.

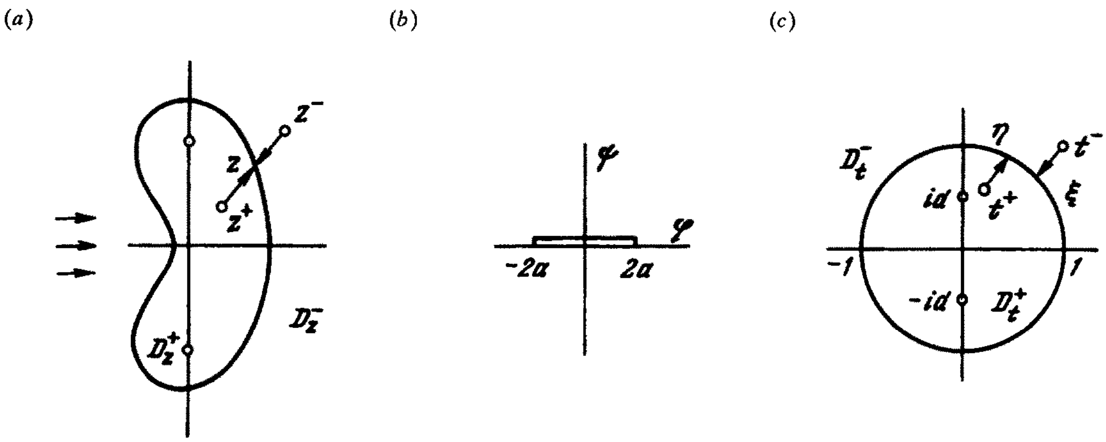

Fig. 1.

where the dimensionless characteristics and independent variables are as follows:

$$z=\frac{Z}{H},\quad \theta^-=\frac{T^--T_0}{T_\infty-T_0},\quad \theta^+=\frac{(T^+-T_0)\lambda^+}{(T_\infty-T_0)\lambda^-}$$
$$p=\frac{KP}{HV_\infty},\quad \mathrm{Pe}=\frac{K_cV_\infty H}{\alpha^-},\quad \mathbf{v}=\frac{\mathbf{V}}{V_\infty},\quad q=\frac{Q}{(T_\infty-T_0)\lambda^-}$$

Here Pe is the Péclet number, D_z^- is the filtration domain, D_z^+ is the domain occupied by the solid body that is being formed, $\Gamma_z=\partial D_z^+=\partial D_z^-$ is its boundary, $\mathbf{V}$ is the filtration velocity, P is the pressure, T^- and T^+ are the temperatures in domains D_z^- and D_z^+, respectively, $\mathbf{n}$ is the normal to Γ_z external to the domain D_z^+, Z_k $(k=1,\ 2)$ are the coordinates of the cooling sources (coolers), K_c is the ratio of the thermal capacities of the liquid and the porous medium, λ^- and λ^+ are the thermal conductivities in domains D_z^- and D_z^+ respectively, α^- is the thermal diffusivity in D_z^-, K is the filtration coefficient, T_0 is the freezing point of the liquid, T_∞ and V_∞ are the temperature and velocity at infinity, Q is the intensity of the coolers, and H is half the distance between the coolers (the freezing columns, which are represented in the z plane by small circles and which are shrunk to points, the heat consumption being conserved).

We introduce the length-scale l and dimensionless parameters h and Pe_*, together with the physical ζ plane that has been normalized with respect to l

$$l=\frac{(2q)^2\alpha^-}{K_cV_\infty},\quad h=\frac{H}{l},\quad \mathrm{Pe}_*=\frac{\mathrm{Pe}}{h},\quad \zeta=\frac{Z}{l}=zh \tag{1.5}$$

The objects Γ_ζ, D_ζ^+, D_ζ^- and ζ_k in the ζ plane correspond to Γ_z, D_z^+, D_z^- and z_k in the z plane.

Equations (1.1) allow the introduction of the complex flow potential $W^-=\varphi+i\psi$ in the usual manner, where $\varphi=-p$, ψ is the stream function, and Eqs (1.3) have a thermal potential $W^+=-\theta^++i\psi^+$, where ψ^+ is the thermal stream function. In the W^- plane the contour Γ_z corresponds to a horizontal cut Γ_w of length $4a$ where a is a parameter to be determined (Fig. 1b). We also introduce the $W_*=\varphi_*+i\psi_*$ plane related to W^-

$$W_*=W^-h,\quad a_*=ah \tag{1.6}$$

We apply the Boussinesq transformation to the first equation in (1.2), which is equivalent to changing from $\theta^-(z)$ to $\theta^-(W_*)$ using as yet unknown function $W_*(z)$, which performs a conformal mapping of the z plane onto the W_* plane. This results in the separation from (1.1)–(1.4) of a closed heat exchange problem between a plate and a uniform flow in the W_* plane

$$\begin{aligned} &\mathrm{Pe}_* \partial\theta^- / \partial\varphi_* = \partial^2\theta^- / \partial\varphi_*^2 + \partial^2\theta^- / \partial\psi_*^2 \\ &\theta^- = 1, \quad |W_*| \to \infty; \quad \theta^- = 0, \quad W_* \in \Gamma_w \end{aligned} \tag{1.7}$$

Because this problem is symmetrical about the real semi-axis in W_*, the heat flux densities $\partial\theta^-/\partial\psi_*$ on the upper and lower sides of the plate Γ_w only differ in sign. If we introduce the function

$$\mu(\varphi_* / 2a_*) = 2a_* |\partial\theta^- / \partial\psi_*|_{W_* \in \Gamma_w}$$

the methods of [5, 6] enable us to reduce problem (1.7) to the boundary integral equation

$$\pi = \int_{-1}^{1} \mu(\xi) K_0(P|t-\xi|) \exp[P(t-\xi)] d\xi, \quad t \in [-1, 1] \tag{1.8}$$

$$q = \int_{-1}^{1} \mu(\xi) d\xi, \quad P = 4a_* q^2 \tag{1.9}$$

The auxiliary parameters a_* and P are obviously related through the unknown function $\mu(\xi)$ to the physical parameter q. If any one of these three parameters is specified the other two are determined.

Without dwelling on the details of the numerical procedure, we now assume that the function $\mu(\xi)$ and, consequently, $\partial\theta^-/\partial\psi_*$ on Γ_w, are known. Then the original problem (1.1)–(1.4) can be formulated as a coupling problem for analytic functions $W_*(\zeta)$ and $W^*(\zeta)$: it is required to find a function $W_*(\zeta)$, analytic throughout D_ζ^- and satisfying the condition

$$dW_* / d\zeta = 1, \quad |\zeta| \to \infty \tag{1.10}$$

and a function $W^+(\zeta)$, analytic throughout D_ζ^+, except at the points ζ_k where it has logarithmic singularities, with boundary conditions on the contour Γ_ζ

$$\operatorname{Im} W_* = \operatorname{Re} W^+ = 0, \quad \frac{dW^+}{d\zeta} = i\frac{\partial\theta^-}{\partial\psi_*}\frac{dW_*}{d\zeta}, \quad \zeta \in \Gamma_\zeta \tag{1.11}$$

It is difficult to solve a coupling problem formulated in this manner because the shape of the contour on which the boundary conditions are specified is unknown. below we shall use a parameterization that is widely used in the theory of ideal fluid jets [7].

We introduce the plane of the auxiliary variable t, in which the contour Γ_ζ corresponds to the unit circle Γ_t centred at the origin. The interior of the circle is denoted by D_t^+ and the exterior by D_t^- (see Fig. 1c).

The conformal mapping $W_*(t^-)$ is given by the Zhukovskii function [7]

$$W_*(t^-) = a_*(t^- + 1/t^-) \tag{1.12}$$

The corresponding mapping $W^+(t^+)$ is found by jet theory methods [8]

$$W^+(t^+) = -\frac{q}{2\pi}\sum_{k=1}^{2} \ln\frac{t^+ - t_k}{1 - t^+\bar{t}_k}$$

The unknown complex parameters t_k correspond to the images of the coolers ζ_k in the domain D_ζ^+. Below we shall restrict ourselves to two arrays of coolers: a tandem array strictly along the stream and a transverse array when the plane passing through the cooler axes is perpendicular to the stream. The corresponding positions of the images ζ_k in the domain D_t^+ will be the points $t_k = \pm d$ and $t_k = \pm id$, where d is an auxiliary parameter to be determined.

We shall perform the detailed calculation for a transverse array of coolers. In this case we have

$$W^+(t^+) = -\frac{q}{2\pi}\ln\frac{(t^+)^2+d^2}{1+d^2(t^+)^2} \tag{1.13}$$

It should be noted that as the point z approaches the contour Γ_z (corresponding to ζ moving towards Γ_ζ) from inside and from outside, in the t plane we respectively obtain images $\eta \in \Gamma_t$ and $\xi \in \Gamma_t$ of that point. In the general case $\xi \neq \eta$ (Figs 1a and c). However the boundary conditions (1.11), with (1.12) and (1.13), enable one to establish the relation $\eta = \eta(\xi)$, which is called the shift [9, 10].

Indeed, condition (1.11) can be written in the form

$$\frac{dW^+}{dt^+}(\eta)F^+(\eta) = i\frac{\partial\theta^-}{\partial\psi_*}[W_*(\xi)]\frac{dW_*}{dt^-}(\xi)F^-(\xi); \quad F^\pm = \left[\frac{d\zeta}{dt^\pm}\right]^{-1}, \quad t^\pm \in D_t^\pm \tag{1.14}$$

from which it follows that

$$\int_{\eta_0}^{\eta}\frac{dW^+}{dt^+}(\eta)d\eta = i\int_{\xi_0}^{\xi}\frac{\partial\theta^-}{\partial\psi_*}[W_*(\xi)]\frac{dW_*}{dt^-}(\xi)d\xi \tag{1.15}$$

The last relation defines the shift $\eta = \eta(\xi)$. We will find its specific form. Suppose

$$\eta = e^{i\beta}, \quad \xi = e^{i\sigma} \tag{1.16}$$

Then taking $\xi_0 = 1$, $\eta_0 = 1$, from (1.12), (1.13) and (1.15) we obtain

$$-\frac{q}{2\pi}\ln\left[\frac{e^{2i\beta}+d^2}{1+d^2e^{2i\beta}}\right] = a_*\int_0^\sigma\frac{\partial\theta^-}{\partial\psi_*}[W_*(e^{i\sigma})](e^{i\sigma}-e^{-i\sigma})d\sigma$$

and after some elementary algebra we obtain

$$\beta(\sigma) = \alpha(\sigma) + \operatorname{arctg}\left[\frac{\sin 2\alpha(\sigma)}{d^{-2}-\cos 2\alpha(\sigma)}\right], \quad \alpha(\sigma) = \frac{\pi}{q}\int_0^\sigma\mu(\cos\sigma)\sin\sigma d\sigma \tag{1.17}$$

Expression (1.17) together with (1.16) gives the requires shift formula.

The junction problem formulated above in the physical plane can now be formulated as a Riemann problem with a shift [9, 10]: it is required to find a function $F^+(t^+)$ analytic throughout D_t^+ and a function $F^-(t^-)$ analytic throughout D_t^-, with a boundary condition on the contour Γ_t

$$F^+[\eta(\xi)] = \Omega(\xi)F^-(\xi), \quad \xi \in \Gamma_t \tag{1.18}$$

and an additional condition at infinity

$$F^-[t^-]\big|_{|t^-|\to\infty} = 1/a_* \tag{1.19}$$

Here $\eta(\xi)$ is the shift function given by (1.16) and (1.17), and according to (1.11) $\Omega(\xi)$ has the form

$$\Omega(\xi) = i\frac{\partial\theta^-}{\partial\psi_*}[W_*(\xi)]\frac{dW_*}{dt^-}(\xi)\left(\frac{dW^+}{dt^+}[\eta(\xi)]\right)^{-1} \tag{1.20}$$

The Riemann problem with shift (Gazeman problem) has been fairly well investigated [9, 10]. Its solution has the form

$$\ln\left[a_*\left(\frac{d\zeta}{dt^-}\right)^{-1}\right] = \frac{1}{2\pi i}\oint_{\Gamma_t}\frac{\varphi(\sigma)}{\xi - t^-}d\xi\bigg|_{\xi=e^{i\sigma}}, \quad t^- \in D_t^- \tag{1.21}$$

$$\ln\left[a_*\left(\frac{d\zeta}{dt^+}\right)^{-1}\right] = \frac{1}{2\pi i}\oint_{\Gamma_t}\frac{\varphi[\beta_{-1}(\sigma)]}{\xi - t^+}d\xi\bigg|_{\xi=e^{i\sigma}}, \quad t^+ \in D_t^+ \tag{1.22}$$

Here $\varphi(\sigma)$ is a complex-valued function satisfying the integral equation

$$\varphi(s) + \frac{1}{4\pi i}\int_{-\pi}^{\pi}\left\{\beta'(\sigma)\left[\operatorname{ctg}\frac{\beta(\sigma)-\beta(s)}{2}+i\right]-\left[\operatorname{ctg}\frac{\sigma - s}{2}+i\right]\right\}\varphi(\sigma)d\sigma =$$
$$= \ln G(s), \quad s \in [-\pi, \pi] \tag{1.23}$$

$$G(\sigma) = \beta'(\sigma)\exp[i(\beta(\sigma)-\sigma)], \quad \sigma \in [-\pi, \pi] \tag{1.24}$$

and $\beta(\sigma)$ and $\alpha(\sigma)$ are represented by formulae (1.17). Having solved Eq. (1.23) it is easy to reconstruct the shape of the ice–soil body using the expression.

$$\frac{d\zeta}{ds} = ia_*\exp\left\{\frac{\varphi(s)}{2} - \frac{1}{4\pi}\int_{-\pi}^{\pi}\varphi(\sigma)d\sigma + i[\Gamma(\varphi(\sigma)|s)+s]\right\} \tag{1.25}$$

obtained from (1.21) using the Sokhotskii–Plemel formula and the notation $\Gamma(\varphi(\sigma)|s)$ for the Hilbert integral of the function $\varphi(\sigma)$ [9]. Finally, in order to return to the z plane, it is necessary to compute

$$h = \frac{1}{2}\operatorname{Im}\{\zeta(t^+)|_{t^+=id} - \zeta(t^+)|_{t^+=-id}\} \tag{1.26}$$

and divide it into $\zeta: z = \zeta/h$. Knowledge of h also enables us to determine the connection between the auxiliary parameter d and the physical parameter Pe. Indeed, from (1.5), (1.6) and (1.9) we have $P = \mathrm{Pe}_* a_*$, from which it follows that

$$\mathrm{Pe} = Ph / a_* \tag{1.27}$$

Thus the original problem (1.1)–(1.4) splits into two problems: to solve Eq. (1.8) with condition (1.9) and to solve Eq. (1.23). The first problem has been thoroughly investigated [1, 5, 11].

The solution given in [5] in the form of a n expansion in Mathieu functions is inconvenient for numerical calculations, but it enables us to draw an important conclusion on the Hölder continuity of the function $[\mu(\cos\sigma)\sin\sigma]$, and consequently, of the function $\alpha'(\sigma)$ (see (1.17)). An efficient numerical algorithm for solving Eq. (1.18) has been proposed [1]. The analysis performed in [11] shows that the function $\alpha'(\sigma)$ satisfies the condition $0 < \alpha'(\sigma) < \infty$ throughout the interval $[-\pi, \pi]$ and, consequently, a function $\alpha_{-1}(\sigma)$ exists that is the inverse of $\alpha(\sigma)$.

Equation (1.23) is an unconditionally and uniquely solvable Fredholm integral equation of the second kind [12]. However, its numerical implementation by the collocation method on a uniform grid [13] encounters difficulties in the most interesting domain of variation of the parameter $d=1-\varepsilon$, $\varepsilon \ll 1$, which corresponds to a thin bridge connecting the ice–soil bodies. This is because the function

$$\beta'(\sigma)=\alpha'(\sigma)\frac{1-d^4}{1-2d^2\cos 2\alpha(\sigma)+d^4}$$

and hence the kernel of the integral equation have an increment of order $1/\varepsilon$, as the contour goes around the points $\sigma=0$, $\sigma=\pm\pi$, while at the same time the right-hand side of (1.23) has an increment of order $\ln\varepsilon$. Nevertheless, a mathematical analysis of the situation that arises enables us to make significant progress towards solving the problem in question and to construct a simple numerical algorithm using the analytical properties of the solution.

2. SMALL P

This limit is fundamental in the study of analytic properties and the construction of an approximate solution of Eq. (1.23). The corresponding functions will have a subscript zero to distinguish them from the general case. The solution of problem (1.8), (1.9) with $P \ll 1$ has the form [14]

$$\mu_0(\xi)=\frac{q}{\pi\sqrt{1-\xi^2}}+Q(P^2\ln 2P);\quad q=\frac{\pi}{\ln(4/P)-\gamma} \tag{2.1}$$

(where γ is Euler's constant). Restricting ourselves to the leading terms in (2.1), from (1.9) and (1.17) we obtain [11]

$$\alpha_0(\sigma)=\sigma \tag{2.2}$$

It is shown in Appendix A that in this case one can directly solve the junction problem in the physical plane and obtain the formula

$$z_0(t^-)=\sqrt{(t^-/d)^2-1}$$

which with $t^-=e^{i\sigma}$ enables one to find the shape of the contour Γ_z. Analysis shows that when $d=1$ the contour has a singular point $z=0$ responsible for the appearance of singular terms in Eq. (1.23) when $\varepsilon\to 0$. Moreover, this formula turns out to be useful both for small P as a test, and in the case of arbitrary P for obtaining points of the contour in the neighbourhood of the bridge when $\varepsilon \ll 1$ (see Appendix B). However, from the point of view of finding the function $\varphi_0(\sigma)$ it is of little use, because the reconstruction of the latter by the inverse method still involves solving the integral equation. The exact solution that has been found for the junction problem nevertheless enables us to hope that a solution of Eq. (1.23) exists in terms of elementary functions.

We will analyse the right-hand side of the equation. To do this we introduce the function $f_0(\sigma)=\ln(G_0(\sigma)/(1-d^4))$. Rewriting it according to (1.17), (1.24) and (2.1), we obtain

$$f_0(\sigma)=-\ln(1-2d^2\cos 2\sigma+d^4)+i\,\mathrm{arctg}\left[\frac{\sin 2\sigma}{d^{-2}-\cos 2\sigma}\right] \tag{2.3}$$

Assertion 1. The function $f_0(\sigma)$ satisfies the relations

$$f_0[(\beta_{-1}(\sigma)]=-[f_0(\sigma)]^*-2\ln(1-d^4) \tag{2.4}$$

$$\Gamma(\mathrm{Re}\, f_0(\sigma)|s)=-2\,\mathrm{Im}\, f_0(s),\quad \Gamma(\mathrm{Im}\, f_0(\sigma)|s)=\frac{1}{2}\mathrm{Re}\, f_0(s) \tag{2.5}$$

where $(\beta_0)_{-1}(\sigma)$ is the inverse function to $\beta_0(\sigma)$ and the asterisk means that d^2 is replaced by $-d^2$.

Proof. Using the representations of sines and cosines in terms of the tangents of half angles and using (1.6), (1.24) and (2.2), we find that

$$1-2d^2\cos 2(\beta_0)_{-1}(\sigma)+d^4=\frac{1-d^4}{1+2d^2\cos 2\sigma+d^4}$$

$$\operatorname{arctg}\left[\frac{\sin 2(\beta_0)_{-1}(\sigma)}{d^{-2}-\cos 2(\beta_0)_{-1}(\sigma)}\right]=-\operatorname{arctg}\left[\frac{\sin 2\sigma}{-d^{-2}-\cos 2\sigma}\right]$$

From this we directly obtain (2.4)

The validity of relations (2.5) follows from the form of the expansion of $f_0(\sigma)$ as Fourier series [15]

$$f_0(\sigma)=\sum_{k=1}^{\infty}\frac{d^{2k}}{k}[2\cos 2k\sigma+i\sin 2k\sigma]$$

and relations for Hilbert integrals [16]

$$\Gamma(\cos k\sigma|s)=-\sin ks,\quad \Gamma(\sin k\sigma|s)=\cos ks$$

We will now construct the solution of the integral equation. We rewrite (1.23) in the equivalent form

$$\varphi_0(s)+\frac{1}{4\pi i}\int_{-\pi}^{\pi}\left[\operatorname{ctg}\frac{\sigma-\beta_0(s)}{2}+i\right]\varphi_0[(\beta_0)_{-1}(\sigma)]d\sigma-$$
$$-\frac{1}{4\pi i}\int_{-\pi}^{\pi}\left[\operatorname{ctg}\frac{\sigma-s}{2}+i\right]\varphi_0(\sigma)d\sigma=f_0(s)+\ln(1-d^4) \tag{2.6}$$

We will seek a solution of (2.6) in the form $\varphi_0(\sigma)=f_0(\sigma)+C_0$, where $C_0=\text{const}$. Denoting by $[L_0\varphi_0](s)$ the integral operator on the left-hand side of (2.6) and using (2.4), (2.5) and the relation

$$\int_{-\pi}^{\pi}f_0(\tau)d\tau=0 \tag{2.7}$$

we have

$$[L_0\varphi_0](s)=f_0(s)+C_0-\frac{1}{2i}\Gamma([f_0(\sigma)]^*|\beta_0(s))-\frac{1}{2i}\Gamma(f_0(\sigma)|s)-\ln(1-d^4)=$$
$$=C_0-\ln(1-d^4)+\tfrac{3}{4}\operatorname{Re}[f_0(s)]-\tfrac{1}{4}\operatorname{Re}[f_0[\beta_0(s)]]-i\operatorname{Im}[f_0[\beta_0(s)]]=$$
$$=C_0-\tfrac{1}{2}\ln(1-d^4)+f_0(s)$$

Comparing the final expression with the right-hand side of (2.6), we find the value of C_0. We finally obtain

$$\varphi_0(\sigma)=f_0(\sigma)+\tfrac{3}{2}\ln(1-d^4) \tag{2.8}$$

We note some properties of the solution obtained. When $\varepsilon \ll 1$ the real part of the function $\varphi_0(\sigma)$ has increments of order $\ln\varepsilon$ in the neighbourhoods of the points $\sigma=0$, $\sigma=\pm\pi$. Here the imaginary part is smoother, being bounded by quantity of order 1. However, it follows from

(1.25) and (2.5) that large values of $|dz_0/dt^-|$ at the points $t=1, -1$ when $\varepsilon \ll 1$ are ensured not only by the real, but also by the imaginary part of $\varphi_0(\sigma)$.

3. SOLUTION OF EQ. (1.23) IN THE GENERAL CASE

It is natural to assume that in the general case the solution $\varphi(\sigma)$ of Eq. (1.23) has the same qualitative properties as $\varphi_0(\sigma)$. In other words, when $\varepsilon \ll 1$ the real part of the function $\varphi(\sigma)$ has a logarithmic singularity at the points $\sigma=0$, $\sigma=\pm\pi$, which also causes the slow convergence of the numerical solution. The imaginary part $\varphi(\sigma)$ is smoother, but together with the real part it causes large values of $|dz_0/dt^-|$ at the points $t=\pm 1$.This means that when the contour of the ice–soil body is reconstructed by direct integration of the function (dz/dt^-), in the neighbourhoods of the given points large errors will be unavoidable, even in the case when the function $\varphi(\sigma)$ is found accurately. Taking all this into account, it makes sense to attempt to isolate the singularity from the integral equation using (2.8).

We shall seek a solution of Eq. (1.23) in the form

$$\varphi(\sigma) = N\varphi_0[\alpha(\sigma)] + \varphi_1(\sigma) \tag{3.1}$$

where N is, for the time being, an arbitrary constant. If $[L\varphi](s)$ denotes the integral operator on the left-hand side of (1.23) acting on the function $\varphi(\sigma)$, and we use

$$\begin{gathered}\beta(\sigma) = \beta_0[\alpha(\sigma)], \quad \beta_{-1}(\sigma) = \alpha_{-1}[(\beta_0)_{-1}(\sigma)] \\ G(\sigma) = G_0[\alpha(\sigma)]g(\sigma), \quad g(\sigma) = \alpha'(\sigma)\exp(i[\alpha(\sigma)-\sigma])\end{gathered} \tag{3.2}$$

then substituting (3.1) into (1.23) we obtain

$$\begin{gathered}[L\varphi_1](s) = \Phi(s) \\ \Phi(s) = \ln g(s) + \ln G_0[\alpha(s)] - N[L\varphi_0](s)\end{gathered} \tag{3.3}$$

We transform $\Phi(s)$ into a more convenient form. Using the substitution $\sigma=\alpha_{-1}(\tau)$, $s=\alpha_{-1}(\tau)$, relation (2.7), and also the fact that the function $\varphi_0(\tau)$ is a solution of the equation $[L_0\varphi_0](t) = \ln G_0(t)$, we find

$$\begin{gathered}\Phi(s) = \ln g(s) + (1-N)\ln G_0[\alpha(s)] + \\ + \frac{N}{2i}\{\Gamma(f_0[\alpha(\sigma)]|s) - \Gamma(f_0(\sigma)|\alpha(s))\} + \frac{N}{4\pi}\int_{-\pi}^{\pi} f_0[\alpha(\sigma)]d\sigma\end{gathered} \tag{3.4}$$

We have thus obtained an integral equation (3.3) for $\varphi_1(\sigma)$ in which the right-hand side contains a provisionally arbitrary parameter N. It is natural to expect a smoother right-hand side to lead to a smoother solution $\varphi_1(\sigma)$. Hence we choose N so that the function $\Phi(\sigma)$ does not contain a singularity of order $\ln\varepsilon$ at the points $\sigma=0, \pm\pi$.

We shall analyse the expression obtained for $\Phi(s)$ from the point of view of its smoothness as $\varepsilon \to 0$. It is clear from (3.2) that the function $\ln g(s)$ has no singularities. The singularities of the function $\ln G_0(s)$ were discussed in Section 1: over a circuit of the contour they have an increment of order $\ln\varepsilon$ in the neighbourhoods of the points $s=0, \pm\pi$. The introduction of $\alpha(s)$ does not change its qualitative behaviour because $\alpha'(0)$, $\alpha'(\pm\pi)$ are finite and do not depend on ε.

The following assertion holds for the difference of the two Hilbert integrals in expression (3.4).

Assertion 2. The function $\Omega(s) = \Gamma(f_0(\sigma)|\alpha(s)) - \Gamma(f_0[\alpha(\sigma)]|s)$ has no singularities of order

$\ln\varepsilon$ for any s. The derivative of $\Omega(s)$ has singularities of order l/ε in the neighbourhoods of $s=0,\ s=\pm\pi$.

The proof is performed by standard Cauchy integral methods [9] using the properties of the function $\alpha(\sigma)$ given in Section 1.

Thus only the function $\ln G_0[\alpha(s)]$ has a singularity of order $\ln\varepsilon$ in the expression for $\Phi(s)$ and, consequently, the smoothest $\Phi(s)$ is achieved by choosing $N=1$

$$\Phi(s)=\ln g(s)+\frac{1}{2i}\{\Gamma(f_0[\alpha(\sigma)]|s)-\Gamma(f_0(\sigma)|\alpha(s))\}+\frac{1}{2\pi}\int_0^\pi \mathrm{Re}\, f_0[\alpha(\sigma)]d\sigma \tag{3.5}$$

We finally arrive at the following problem for $\varphi_1(\sigma)$: find a solution of Eq. (3.3) with the same integral operator as in (1.23), but with a right-hand side of the form (3.5). The solution of Eq. (1.23) is then constructed directly from (3.1), where the constant N must be set equal to unity.

Appendix B gives the most important properties of the numerical implementation of finding $\varphi_1(\sigma)$ and the shapes of the ice–soil body. Note that the conclusions and formulae for the transverse array of the coolers obtained both above and in the Appendices also hold for the tandem array if one formally replaces id by d and d^2 by $(-d^2)$.

We shall make some remarks on the limit as $\varepsilon\to 0$. This case corresponds to an infinitely narrow bridge. We shall call it the critical regime. The corresponding values of the physical characteristics are also called critical. Because it is impossible to put $\varepsilon=0$ in the calculation, we put $\varepsilon=10^{-5}$. The thickness of the bridge was found to be ≈ 0.01.

4. DISCUSSION OF THE RESULTS

A series of calculations was performed to determine the shape of the ice–soil body formed by a system of two coolers with a given intensity q (see Figs 2 and 3, where $q=0.996$ (a) and $q=7.18$ (b)) and a given Pe (see Figs 4 and 5 with $\mathrm{Pe}=0.5$ (a) and $\mathrm{Pe}=2$ (b)). Because the process is symmetrical about the x axis, only the upper halves of the bodies are shown.

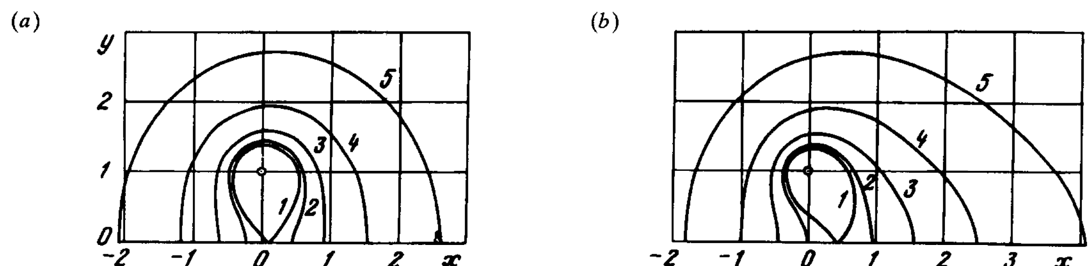

Fig. 2.

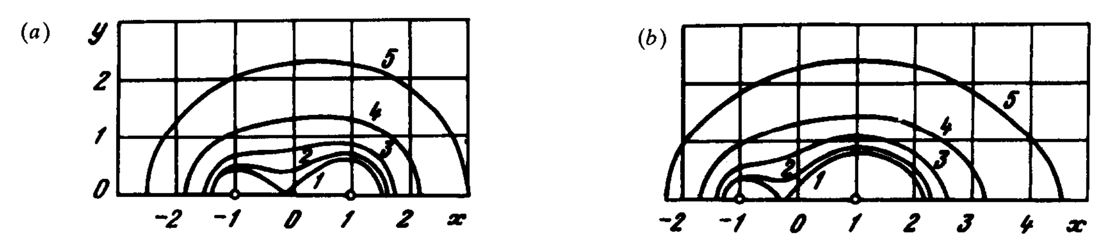

Fig. 3.

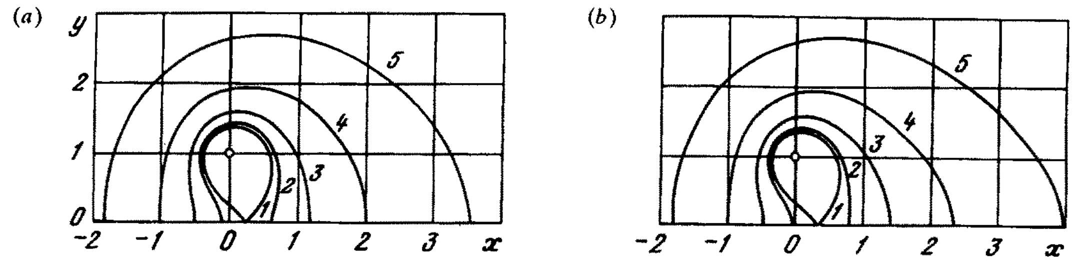

Fig. 4.

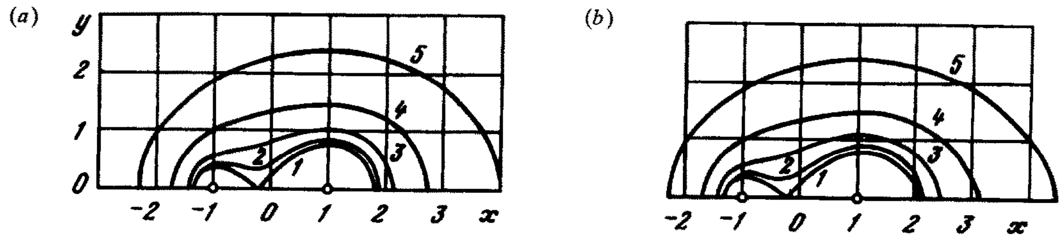

Fig. 5.

Table 1 shows the values of physical parameters for the different curve numbers. It is clear that in nearly-critical regimes the body has a complex form which depends significantly on the orientation of the pair of coolers relative to the free flow. At the same time, in critical regimes the local behaviour of the contour of the ice–soil body in the neighbourhood of the bridge point is universal: here the contour has a cross-shaped right-angled intersection. In far-from-critical regimes the shape of the body depends only slightly on the orientation of the coolers and is close to that of a body formed by a single cooler of the same total power $2q$ [1].

Figure 6 shows the behaviour of the boundary of the ice–soil body in the neighbourhood of the bridge as the critical regime is approached for the transverse array of the coolers. (Here curve 1 corresponds to curve 1 in Fig. 2b.) It is clear that this boundary is always a smooth curve, but its curvature at the intersection with the $y=0$ axis increases without limit as the critical regime is approached. Only in the limit of the critical regime is there a break point—it is the point at which separate ice–soil bodies link up.

Figure 7 shows the dependence of the cross-sectional area of the ice–soil body S on the power of the coolers q for various Péclet numbers. The dashed lines are the corresponding $S_{cr}(q)$ curves. Henceforth the unprimed numbers are for the transverse array of coolers, and the primed numbers are for the tandem array. It is clear that for small Pe the cross-sectional

Table 1

N	Pe				q			
	Fig. 2(a)	2(b)	3(a)	3(b)	4(a)	4(b)	5(a)	5(b)
1	0,1000	10.32	0,0976	8,196	1,77	3.25	1,84	3.57
2	0,0991	10,18	0,0928	7,846	1,80	3,36	1,90	3,64
3	0,0950	9.530	0.0784	6,726	1.97	3.75	2.03	3.94
4	0.0797	7.630	0,0590	5,146	2,24	4.38	2.29	4.49
5	0.0596	5.510	0.0395	3,485	2.75	5.39	2.77	5.47

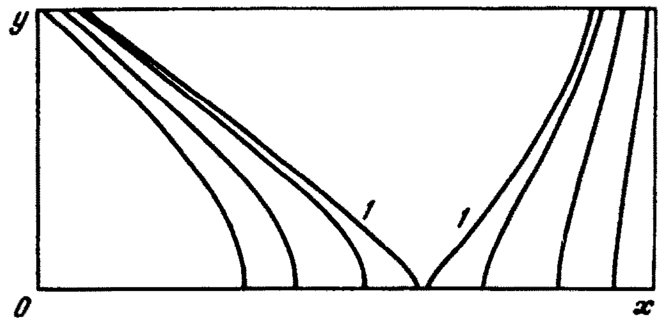

Fig. 6.

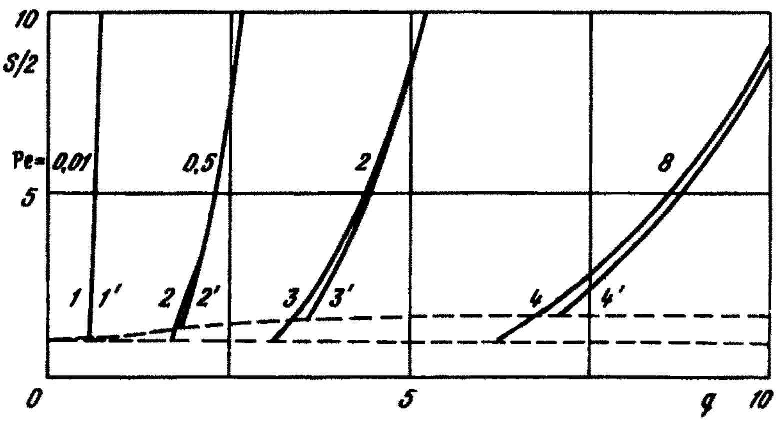

Fig. 7.

area of the ice–soil body does not depend on the orientation of the coolers even in near-critical regimes. For large Pe with the same cooling power the transverse array gives a somewhat larger cross-sectional area of the ice–soil body. At the same time, the tandem array gives a larger critical cross-sectional area S_{cr} (Fig. 8, curves 2 and 2′) because the corresponding critical value q_{cr} is larger (curves 1 and 1′). The same graph shows that in the case of the transverse array of the coolers the value of S_{cr} is approximately constant and does not depend on Pe, although calculations have shown that the shape of the ice–soil body changes considerably up to the last calculated value of $\mathrm{Pe} \approx 100$. In the case of tandem array the quantity S_{cr} undergoes significant growth in the range $\mathrm{Pe} \in [0, 4]$, mostly at small Pe, and is then practically constant. Here the shape of the leading body already stabilizes at $\mathrm{Pe} = 0.85$. The shape of the rear body continues to change as Pe increases in the neighbourhood of the rear point.

It is interesting to compare the results obtained with those of [4] where a similar problem was considered with the assumption that $\mathrm{Pe} \gg 1$ and that the heat transfer along the stream lines could be neglected. As has already been noted [11], a similar approach applied to a single body gives a distorted picture in the neighbourhood of the rear point of the body. (Instead of a smooth contour one obtains a sharp cusp.) In the case of two coolers in the transverse array, formulae previously obtained [4] for the critical regime give a distorted picture in the neighbourhood of the point of contact—all the angles made by the contours of the body with the x axis are different from those obtained by calculation.

However, the linking condition for separate bodies obtained in [4] is of fundamental interest. From the solution of the problem for a single ice–soil body formed by a pair of coolers in the transverse array, with the assumption given above, the condition

$$q > 3^{-3/4}\sqrt{8\pi \mathrm{Pe}} \tag{4.1}$$

was obtained.

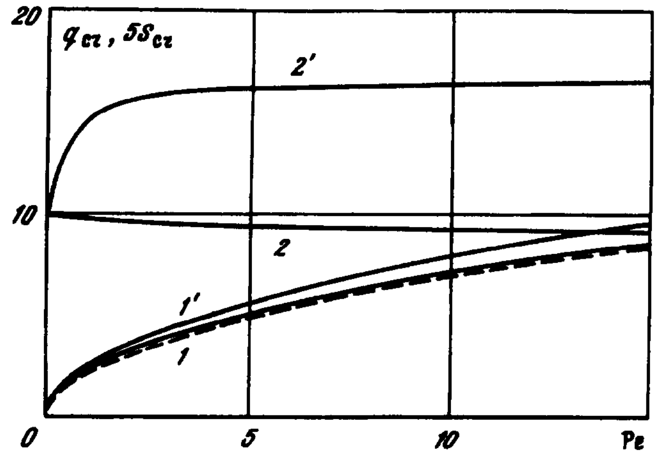

Fig. 8.

It is clear that the expression on the right of (4.1) is an estimate of the dependence $q_{cr}(\mathrm{Pe})$, and the graph shown in Fig. 8 demonstrates that this estimate is fairly accurate (the dashed line).

At the same time, it is clear by construction that the approximate condition (4.1) and the more accurate condition

$$q > q_{cr}(\mathrm{Pe}) \tag{4.2}$$

are in fact "non-separation" conditions, because they were obtained under the assumption that there was a single ice–soil body. We then have the problem of whether the "non-separation" condition holds as a linking condition.

This problem can be analysed completely when the problem of two separate bodies is solved. But one can perform a partial analysis, at least for large Pe. In this case it is natural to assume that an insulating wall between the ice–soil bodies (the plane of symmetry of the flow) has no effect on the shapes of the bodies when their sizes are small compared to the distance from the cooler to the wall. One can consequently take each of the bodies to be single and obtain their shape by well-known methods [1].

Figure 9 shows the shape of a single ice–soil body for $\mathrm{Pe}=5$ when $q=q_{cr}=5.01$ (the solid curve) and separate bodies when $q=5.21>q_{cr}$ (the dashed curve). Notwithstanding the fact that condition (4.2), and all the more so (4.1), are satisfied, it is clear that the separate bodies are far from linking. One could imagine that the assumption adopted holds. However, calculations show that if one takes yet another insulating wall, symmetrical to the first with respect to the position of the cooler (flow past a linear array of coolers, to be published), the shape of the ice–soil body is almost the same as that of the single body shown in Fig. 9.

Thus we arrive at the conclusion that, at least for large Pe, one cannot use the condition proposed previously [4] as a linking criterion for ice–soil bodies, nor any other criterion based on solving the problem for a united body.

APPENDIX A

The solution of the junction problem for small P has already been published in compressed form in conference proceedings [17]. Here we give a detailed account.

We restrict ourselves to the leading term in (2.1). We have

$$\frac{\partial\theta^-}{\partial\psi_*}=\frac{q}{2a_*\pi}\left[1-\left(\frac{\varphi_*}{2a_*}\right)^2\right]^{-1/2} \tag{A1}$$

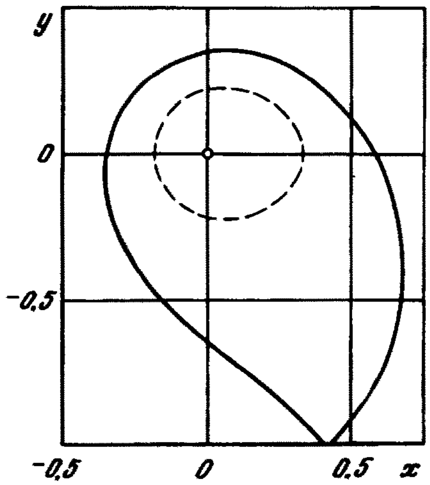

Fig. 9.

Here the root is positive at the upper edge of the cut in W_*, and negative at the lower edge.

The junction condition can be written in the form (1.11). After changing from ζ to z and using (A1) this gives

$$\frac{iq}{2a_*\pi}\left[1-\left(\frac{W_*}{2a_*}\right)^2\right]^{-1/2}\frac{dW_*}{dz}=\frac{dW^+}{dz},\quad z\in\Gamma_z \tag{A2}$$

It can be shown that after multiplying by dz the left-hand side of this relation can be written in the form

$$i\left(\frac{q}{\pi}\right)\left[1-\left(\frac{W_*}{2a_*}\right)^2\right]^{-1/2} d\left(\frac{W_*}{2a_*}\right)=\pm\left(\frac{q}{\pi}\right)d\ln\chi(W_*) \tag{A3}$$

if one takes the function $\chi(W_*)$ to be

$$\chi(W_*)=it^-=i\left[\left(\frac{W_*}{2a_*}\right)+\sqrt{\left(\frac{W_*}{2a_*}\right)^2-1}\right] \tag{A4}$$

Thus the condition on Γ_z acquires the even simpler form

$$\left(-\frac{q}{\pi}\right)\frac{d}{dz}\ln\chi=\frac{dW^+}{dz},\quad z\in\Gamma_z \tag{A5}$$

We will investigate the singularities of the functions on the right- and left-hand sides of (A5). In the domain D_z^+ the function dW^+/dz obviously has the singularities

$$\left.\frac{dW^+}{dz}\right|_{z\to\pm i}=-\frac{q}{2\pi}\frac{1}{z\mp i}+O(1) \tag{A6}$$

Comparing the domain D_z^- with the image domain of the function $\chi(z)$, or using (A4) with the domain D_t^- we see that $\chi(z)$ is regular and non-zero throughout D_z^-. Moreover, taking (1.5), (1.6) and (1.10) into account, we obtain from (A4) that

$$\left.\frac{d\chi}{dz}\right|_{|z|\to\infty}=\frac{i}{a};\quad \chi(z)|_{|z|\to\infty}=\frac{i}{a}z+O(1) \tag{A7}$$

The function $d\ln\chi/dz$ consequently has no singularities in D_z^-, and at the point $|z|\to\infty$ has a first-order zero

$$\left.\frac{d}{dz}\ln\chi\right|_{|z|\to\infty}=\frac{1}{z} \tag{A8}$$

Thus the functions on the right- and left-hand sides of (A5) have no singularities other than poles. One can therefore introduce a single meromorphic function $\omega(z)$

$$\omega(z)=\begin{cases}-(q/\pi)d\ln\chi/dz, & z\in D_z^-,\\ dW^+/dz, & z\in D_z^+,\end{cases} \tag{A9}$$

that has two simple poles in the z plane and regular behaviour at infinity. We know [7] that such a function can be reconstructed from its singularities: $\omega(z)=A(z+B)/(z^2+1)$.

The complex constants A and B are determined from (A6) and (A8): $A=-q/\pi$; $B=0$. Finally, we obtain

$$\omega(z)=-(q/\pi)z/(z^2+1) \tag{A10}$$

Now, taking (A9) into account, we find that

$$W^+(z)=-\frac{q}{2\pi}\ln[(z^2+1)C^+],\quad \chi(z)=C^-\sqrt{z^2+1} \tag{A11}$$

The constants of integration C^+ and C^- are determined from (A7), (A9) and (A10) taking (1.13) into account

$$C^-=i/a,\quad C^+=d^2=1/a^2 \tag{A12}$$

Substituting them into (A11) and using (A5) and (1.13), one can determine the shape of the contour Γ_z

$$z=\sqrt{\left(\frac{t^-}{d}\right)^2-1},\quad t^-=e^{i\sigma};\quad z=\sqrt{\frac{1-d^4}{d^4+(d/t^+)^2}},\quad t^+=e^{i\beta} \tag{A13}$$

Using the shift formula (1.17) and taking (2.2) into account, one can verify that body expressions in (1.13) give one and the same family of contours.

The relation between the auxiliary parameter d and the physical parameter Pe for given P (which is equivalent to a given q) is established from (1.27) taking (1.6) and (A12) into account

$$\mathrm{Pe}=Pd \tag{A14}$$

The case $d=1$ corresponds to the critical regime, and from the second relation in (2.1) and (A14) one can determine $q_{cr}(\mathrm{Pe})$

$$q_{cr}=\frac{\pi}{\ln(4/\mathrm{Pe})-\gamma} \tag{A15}$$

One should however bear in mind that the representation of $q(P)$ in (2.1) only works well when $P\leqslant 0.1$. Consequently the asymptotic formula in (A15) only holds when $\mathrm{Pe}\leqslant 0.01$.

APPENDIX B

Because high accuracy is required when calculating nearly critical regimes, the numerical implementation of the algorithm described above requires special attention. We shall describe the more important points.

The integral equation (3.3) for the complex function $\varphi_1(\sigma)$ is transformed into a system of two integral equations for the real functions $\mathrm{Re}\varphi_1(\sigma)$, $\mathrm{Im}\varphi_1(\sigma)$. The resulting system is solved by the collocation method on a non-uniform grid. The matrix obtained in well-conditioned because the form of the system ensures predominance of the diagonal elements. The standard Gaussian elimination method with partial choice of leading element is used to invert it.

The need for a non-uniform grid is due to the fact that the derivative of $\Phi(\sigma)$, which is the right-hand side of the integral equation for $\varphi_1(\sigma)$, has singularities of the type $1/\varepsilon$ at the points $\sigma=0$, $\sigma=\pm\pi$ (see Assertion 2). These singularities are enough to introduce a significant (~1%) error into the shape of the ice–soil body at $d=0.995$ (when the thickness of the bridge is still ~0.2) if, for example, the grid has 128 nodes in the interval $[0, \pi]$. Applying a non-uniform grid concentrated around $\sigma=0$, $\sigma=\pm\pi$ for the transverse array and around $\sigma=\pm\pi/2$ for the tandem array of the coolers one can get to $d=0.99999$ with 100 nodes and a corresponding error of $<0.5\%$ (the bridge thickness ≈ 0.01). The numerical error is estimated from the closeness of the resulting contour to closure.

A special approach is used to compute the integrals on the right-hand side of (3.5). The latter, using integration by parts, is reduced to the form

$$\mathrm{Re}\,\Phi(s)=\ln\alpha'(s)-\frac{1}{2\pi}\int_0^\pi R(\sigma,s)\,\mathrm{Im}\{f_0'[\alpha(\sigma)]\}\alpha'(\sigma)d\sigma-\frac{1}{2\pi}\int_0^\pi[\alpha_{-1}(\sigma)-\sigma]\,\mathrm{Re}\{f_0'(\sigma)\}d\sigma$$

$$\mathrm{Im}\,\Phi(s)=[\alpha(\sigma)-\sigma]+\frac{1}{2\pi}\int_0^\pi J(\sigma,s)\,\mathrm{Re}\{f_0'[\alpha(\sigma)]\}\alpha'(\sigma)d\sigma$$

where

$$R(\sigma,s)=\ln\left|\sin\frac{\sigma-s}{2}\sin\frac{\sigma+s}{2}\Big/\left[\sin\frac{\alpha(\sigma)-\alpha(s)}{2}\sin\frac{\alpha(\sigma)+\alpha(s)}{2}\right]\right|$$

$$J(\sigma,s)=\ln\left|\sin\frac{\sigma-s}{2}\sin\frac{\alpha(\sigma)+\alpha(s)}{2}\Big/\left[\sin\frac{\alpha(\sigma)-\alpha(s)}{2}\sin\frac{\sigma+s}{2}\right]\right|$$

Using (2.3) we see that due to the presence of the derivative of f_0 all the functions in the integrand behave badly in the neighbourhoods of $\sigma=0$, $\sigma=\pm\pi$ and standard methods of numerical integration produce large errors. The following technique is therefore used. Using the fast Fourier transform [18] the functions $R(\sigma,s)$, $J(\sigma,s)$ and $[\alpha_{-1}(\sigma)-\sigma]$ are expanded at each point s in terms of $\cos[k\alpha(\sigma)]$, $\sin[k\alpha(\sigma)]$ and $\sin(k\sigma)$, respectively. (The latter expansion does not depend on s and was naturally only found once.) It is clear that the behaviour of the expanded functions is determined only by the function $\alpha(\sigma)$, which is always smooth. Hence the expansion coefficients decrease rapidly and are themselves represented by series that are fairly exact. (In the calculations n was taken to be 128.) After this all the integrals were evaluated exactly [19].

As has already been remarked, when $\varepsilon \ll 1$ one must consider the problem of obtaining points on the contour of the ice–soil body near the bridge. It was assumed that the violation of the conformality of $\zeta(t^-)$ at the points $t^-=\pm1$ was only due to the contribution of the function $\varphi_0[\alpha(s)]$ to (3.1). Then, from (1.25) and taking into account (A13), (2.3), (2.8) and (3.5), the formula

$$\frac{d\zeta(e^{is})}{ds}=\left.\frac{d\zeta_0(e^{i\sigma})}{d\sigma}\right|_{\sigma=\alpha(s)}\cdot\alpha'(s)\exp\left[\frac{1}{2}\varphi_1(s)+\frac{i}{2}\Gamma(\varphi_1(\sigma)|s)\;-\frac{1}{4\pi}\int_{-\pi}^{\pi}\varphi_1(\sigma)d\sigma-\Phi(s)\right]$$

is obtained, which was also used for integration in the neighbourhoods of $s=0$ and $s=\pm\pi$. The rapid convergence of the integration when the step size is reduced enables us to assert that the assumption was correct, and hence that locally the neighbourhood of the bridge the contour of the ice–soil body in general has the same singularity as for small P.

To compute the normalization factor h one must take into account that $\zeta(t^+)$ gives a conformal mapping of D_t^+ onto D_ζ^+, and after reconstructing the contour of the ice–soil body its boundary values are known: $\zeta(t^+)|_{t^+d=e^{i\sigma}} = \zeta(t^-)|_{t^-=e^{i\beta^{-1}(\sigma)}}$. Then, as is well known, the value of the function at any internal point can be obtained with the help of the Cauchy integral [7]. In particular, for $t^+ = \pm id$ we obtain

$$\zeta^+(\pm id) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\zeta[e^{i\beta^{-1}(\sigma)}]}{1\mp ide^{i\sigma}}\,d\sigma,$$

after which it is easy to find h.

REFERENCES

1. KORNEV K. G. and MUKHAMADULLINA G. I., On the influence of flow in a porous medium on the equilibrium shape of a body formed by artificial freezing. *Inzh.-Fiz. Zh.* **61**, 6, 980–985, 1991.
2. NASONOV I. D., *The Freezing of Filtering Rocks*. Nedra, Moscow, 1968.
3. SADOVSKI A. V. and DORMAN Y. A., Artificial freezing and freezing of soil at the construction. In *Proceedings of the 2nd International Symposium on Ground Freezing*, pp. 862–887. Trondheim, Elsevier, Amsterdam, 1982.
4. MAKSIMOV V. A., The shape of merging ice–soil bodies formed by a battery of freezing rods. In *Some Problems in Rock Mechanics*, pp. 110–109, Mosk. Gorn. Inst., Moscow, 1968.
5. RVACHEV V. L., The pressure on an elastic half-space from a strip punch. *Prikl. Mat. Mekh.* **20**, 2, 248–254, 1956.
6. CHUGUNOV V. A. and KORNEV K. G., The dynamics of ice barriers during the freezing of filtering rocks. *Inzh.-Fiz. Zh.* **51**, 2, 305311, 1986.
7. LAVRENT'YEV M. A. and SHABAT B. V., *Methods of the Theory of Functions of a Complex Variable*. Nauka, Moscow, 1973.
8. BIRKHOFF G. and SARANTONELLO E., *Jets, Wakes and Cavities*. Mir, Moscow, 1964.
9. GAKHOV F. D., *Boundary-value Problems*. Fizmatgiz, Moscow, 1963.
10. LITVINCHUK G. S., *Boundary-value Problems and Singular Integral Equations with Shifting*. Nauka, Moscow, 1977.
11. KORNEV K. G. and CHUGUNOV V. A., Determining the equilibrium shape of a body formed by the freezing of flow in a porous medium. *Prikl. Mat. Mekh.*, **52**, 6, 991–996, 1988.
12. MUSKHELISHVILI N. I., *Singular Integral Equations*. Nauka, Moscow, 1968.
13. BELOTSERKOVSKII S. M. and LIFANOV I. K., *Numerical Methods in Singular Integral Equations and their Applications to Aerodynamics, Elasticity Theory and Electrodynamics*. Nauka, Moscow, 1985.
14. ALEKSANDROV V. M. and KOVALENKO Ye. V., *Problems in the Mechanics of Continuous Media with Mixed Boundary Conditions*. Nauka, Moscow, 1986.
15. PRUDNIKOV A. P., BRYCHKOV Yu. A. and MARYCHEV O. I., *Integrals and Series*. Nauka, Moscow, 1981.
16. PYKHTEYEV G. N., *Exact Methods for Calculating Cauchy-Type Integrals*. Nauka, Novosibirsk, 1980.
17. KORNEV K. and MUKHAMADULLINA G., Mathematical theory of freezing for flow in porous media. In *Flow through Porous Media:: Fundamentals and Reservoir Engineering Applications*, pp. 60–63. Proc. Int. Conf., Moscow, Inst. Probl. Mekh. Ross. Akad. Nauk, Moscow, 1992.
18. SAMARSKII A. A. and NIKOLAYEV Ye. S., *Methods of Solving Grid Equations*. Nauka, Moscow, 1978.
19. GRADSHTEIN I. S. and RYZHIK I. M., *Tables of Integrals, Sums, Series and Products*. Fizmatgiz, Moscow, 1962.

Translated by R.L.Z.